2/11/20

Mixtures of r.v.'s

Part 1: Discrete mixtures of discrete distributions (last)

Part 2: Discrete mixture of continuous r.v.'s

Example: Suppose X is a 80%/20% mixture of W/Y where $W \sim Exp(\theta = 50)$
$Y \sim Exp(\theta = 1000)$

Q: $Var(X) = ?$

A: Method 1: (Mix)

$$Var(X) = \underbrace{E[X^2]}_{mix} - (\underbrace{E[X]}_{mix})^2$$

$$E[X^2] = .8 \cdot E[W^2] + .2\, E[Y^2]$$
$$= .8\,(2 \cdot 50^2) + .2\,(2 \cdot 1000^2) = 404000$$

$$E[X] = .8 \cdot E[W] + .2\, E[Y] = .8(50) + .2(1000) = 240$$

$$\therefore Var(X) = 404000 - 240^2 = 346{,}400$$

Method 2: (Law of Total Variance)

Let I = "indicator" r.v. ("W" or "Y")

$$Var(X) = E[Var(X \mid I)] + Var(E[X \mid I])$$

I	E[X | I]	Var(X | I)	Pr
"W"	50	50^2	.8
"y"	1000	1000^2	.2

$$\therefore \ Var(X) = \Big[\overbrace{\left(50^2(.8)+1000^2(.2)\right) - \left(50(.8)+1000(.2)\right)^2}^{var(E[X \mid I])}\Big]$$

$$+ \Big[\overbrace{50^2(.8) + 1000^2(.2)}^{E[Var(X \mid I)]}\Big]$$

$$= 346{,}400$$

See Handout for another example
(Next Page)

A portfolio consists of 2500 independent one-year insurance policies. There can be at most one claim made during the year, and for each insured the probability of a claim in the year is 0.02. If a claim is made, the benefit will be exponentially distributed with mean 400. Determine the variance of the total amount of benefits paid in the year.

(A) 5,000,000 (B) 7,840,000 (C) 8,000,000 (D) 12,840,000 (E) 15,840,000

$T = \sum_{1}^{2500} Y_i$ $\quad$ Y is a 98%/2% mixture of $\overset{\text{zero}}{0}/X$

where $X \sim Exp(\theta = 400)$

$Var(T) = 2500 \cdot Var(Y)$

Method 1: "Mixing" $\quad Var(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = .98(0) + .02(2 \cdot 400^2) = \cancel{3200}\ 6400$

$E[Y] = .98(0) + .02(400) = 8$

$\therefore Var(Y) = \cancel{3200}\ 6400 - 8^2 = \cancel{3136}\ 6336$

$\therefore Var(T) = 2500(\cancel{3136}\ 6336) = 15,840,000$

Method 2:

I	$E[Y \mid I]$	$Var(Y \mid I)$	Pr
0	0	0	.98
"X"	400	400^2	.02

$Var(Y) = \underbrace{[(0^2(.98) + 400^2(.02)) - (0(.98) + 400(.02))^2]}_{Var(E[Y \mid I])}$

$+ \underbrace{[0(.98) + 400^2(.02)]}_{E[Var(Y \mid I)]} = 6336$

$Var(T) = 15,840,000$

Part 3: Continuous mixture of discrete distribution

Example: $(N \mid \Lambda = \lambda) \sim P(\lambda)$ where $\Lambda \sim \Gamma(\alpha, \theta)$

shortcut $\to$ $N \mid \Lambda \sim P(\Lambda)$

Q: Can we determine the unconditional distribution of N?

Side Remark: The gamma function (see P. 2 of Tables)

is $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt \qquad (\alpha > 1)$

uv - $\int$vd

$u = t^{\alpha-1} \qquad v = -e^{-t}$

$du = (\alpha-1) \cdot t^{\alpha-2} \cdot dt \qquad dv = e^{-t}\, dt$

$$= \cancel{-t^{\alpha-1} e^{-t} \Big|_0^\infty} \to 0 + \int_0^\infty (\alpha-1) \cdot t^{\alpha-2} \cdot e^{-t}\, dt$$

$$= (\alpha-1) \cdot \int_0^\infty t^{\alpha-2} \cdot e^{-t}\, dt$$

$$= (\alpha-1) \cdot \Gamma(\alpha-1)$$

$$\Gamma(\alpha) = (\alpha-1) \cdot \Gamma(\alpha-1)$$

Example: $$\frac{\Gamma(\alpha+5)}{\Gamma(\alpha)} = \frac{(\alpha+4)\cdot(\alpha+3)\cdot(\alpha+2)\cdot(\alpha+1)\cdot \alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

Generally, $$\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha \cdot (\alpha+1) \cdots (\alpha+k-1)$$